A Hierarchy of Strategies for Solving Linear Equations

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This paper describes an investigation of students' strategies for solving linear equations. The assessment techniques used for investigating algebra paralleled those used for investigating number in the New Zealand Numeracy Development Projects (NDPs). In this study of 621 Year 7 to Year 10 students oral interviews were used to investigate the strategies that students used to solve equations. Rasch analysis was used to investigate item difficulty and student ability, and then the strategies associated with each question were examined. The data suggest that there is a hierarchy of sophistication of strategies. Many students were unable to solve a lot of the equations as they were restricted to less sophisticated strategies. The most sophisticated strategy of solving equations by performing transformations was understood by very few students.

A new curriculum for New Zealand schools was introduced in 2007 (Ministry of Education, 2007). The learning area of Mathematics and Statistics is now divided into three strands rather than the previous six, with Number and Algebra being one strand. The implementation of the curriculum in Number up to Year 10 of schooling is guided by the Numeracy Development Projects (NDPs), which provide a framework for children's development in number. The achievement objectives of the new curriculum are grouped to reflect the structure of the Number Framework (Ministry of Education, 2003), which details the number strategies that students use and the number knowledge required for these strategies. At the lower levels of the new curriculum the Number and Algebra achievement objectives are divided up into *number strategies, number knowledge, equations and expressions,* and *patterns and relationships*. The integration of number and algebra into one strand follows debate within the mathematics education community in New Zealand and within international research (see for example Carraher & Schiemann (2007) and Kieran (1992)) as to what constitutes algebraic thinking.

The NDPs have been successful at raising the achievement of New Zealand children in the strand of Number (G. Thomas & Tagg, 2007) and various initiatives are currently underway to extend the projects into early algebra. This study takes a similar approach to the NDPs by specifically examining students' strategies for solving linear equations through the use of oral interviews.

Children's Strategies

Students have struggled with introductory algebra for a long time (Cockcroft, 1982) and teachers have little to guide them in designing programmes of learning. Little is known about the strategies that students use to solve equations or how these strategies are related to conceptual development. A useful summary of strategies used by students is, however, provided by Kieran (1992), who describes the use of known basic facts, counting techniques, guess and check, cover-up, working backwards and formal operations.

The difficulties that students experience, related to their use of algebraic strategies, are well documented. Because much arithmetic in schools is presented as a computation ready to complete, e.g. $3 + 5 =$, and because pressing the equals button on a calculator performs a calculation on whatever has been entered, children understand equals as meaning *compute now* rather than *is equivalent to* (Booker, 1987; Booth, 1988). Linchevski (1995)

In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1). Palmerston North, NZ: MERGA.

states that in the transition from arithmetic to algebra children need to move from a unidirectional view of the equals sign to a multidirectional one.

Closely related to this is the use of an equation as a process, rather than an object that can be operated on (Sfard, 1991). Children initially see equations as the description of an arithmetic process, e.g. $2 \times 3 + 4 = x$, and when presented with an equation to solve, e.g. $2x + 4 = 10$, they also see it as the description of an arithmetic process with *guess and check* as a natural way of finding *x*. Even the more sophisticated strategy of solving the equation by working backwards may result from a view of equations as processes, yet this is often not revealed until students encounter equations of the kind $2x + 4 = 3x - 6$. It is no longer possible to regard the equation as the description of a process giving a result, and it is essential to view the equation as an object to be acted upon in order to solve it (Sfard & Linchevski, 1994). Herscovics & Linchevski (1994), however, have presented data to show that many children revert to the strategy of guess and check to solve equations of this type. The issue of operating on unknowns is another perspective on why equations with unknowns on both sides cause so many difficulties. Booker (1987) suggests that it is the shift from manipulation of numbers in order to solve for an unknown, to the manipulation of unknowns themselves that marks the entry into algebra proper.

Children's stage of numeracy is likely to be important for their understanding of expressions and equations (Irwin, 2003). Equations of the form $x + 3 = 7$can be solved by advanced counters through guess and check, but can be solved much more easily by part-whole thinkers able to visualise 7 as $4 + 3$. It can be argued that to solve equations such as $3x = 15$ requires multiplicative thinking in order for a child to do more than simply follow prescribed algorithms. Furthermore equations of the kind $2x + 3 = 11$ might require an understanding of numbers beyond simple additive part-whole or multiplicative part-whole thinking. A large-scale study by Warren (2003) found that the majority of students leave primary school with a limited awareness of mathematical structure and of arithmetic operations as general processes.

It is important to appreciate that algebra is much more than the manipulation of symbols and that it may be viewed as the symbolising of general numerical relationships and mathematical structures and operating on those structures (Kieran, 1992). Thomas and Tall (2001) describe the long-term cognitive development of symbolic algebra as from operational procepts of arithmetic, to evaluation processes in generalized arithmetic, to manipulation of procepts in manipulation algebra and finally defined concepts in axiomatic algebra. They suggest that each transition involves considerable cognitive reconstruction that acts as a barrier for many. Bednarz, Kieran, and Lee (1996) comment that algebra in schools is often reduced to rules for transforming and solving equations. As rule based algebra can be regarded as based on defined concepts it is not surprising that so many students experience confusion when a rule based approach is used.

Within the NDPs rather than focusing on simply whether children successfully solve arithmetic problems, the strategies that children use to solve the problems are documented. Children are then grouped for instruction according to their most sophisticated arithmetic strategy rather than the difficulty of arithmetic problem (Ministry of Education, 2008). The teaching of solving equations has traditionally focussed on the difficulty of the equations presented to students rather than the strategies that students are actually using to solve them. To move to an approach more consistent with the NDPs more information is needed about how students solve equations.

Methodology

This study of students' understanding of strategies for solving equations related the strategies used to item difficulty and student ability. A structured diagnostic interview was administered to individual students by the researcher or the students' classroom teacher. The students' responses were coded and then analysed making use of Rasch Analysis (Wright & Masters, 1982).

Subjects

The study took place in a two intermediate schools (Years 7 and 8), two high schools (Years 9 and 10) and one college (Years 7 to 9). There was no attempt at representative sampling but instead the aim was to collect data from a wide range of students. The interview was administered to a total of 621 students in Year 7 (n=196), Year 8 (n=43), Year 9 (n=245) and Year 10 (n=137). Clearly Year 8 students are under-represented but this is ameliorated by the fact that interviews took place throughout the school year so students at the beginning of Year 9 and end of Year 7 were included. In the two schools where there was streaming all classes from each year level were included and in all schools no students were excluded on the basis of ability.

Diagnostic Interview

The diagnostic interview was developed in a previous study (Linsell, McAusland, Bell, Savell, & Johnston, 2006) and was guided by the literature on students' strategies for solving equations (Herscovics & Linchevski, 1994; Kieran, 1992).

The interview consisted of a series of increasingly complex equations, which the students were asked to solve with an explanation of their thinking. This work is part of a larger study, which also gathered further information but in this paper it is only the students' responses to symbolic equations that is reported on.

The questions were presented on cards so that the more difficult questions could be omitted as required without suggesting to the student that they were not coping. Each question was read to the student to minimise the impact of difficulties with reading symbolic equations. Calculators and pencil and paper were available for the students to use, but it was stressed that they could use whatever method they chose.

The interviewer recorded what the student did and said and then classified the strategy used according to Table 1.

Table 1
Classification of strategies for solving equations

Code	Strategy
0	Unable to answer question
a	Known basic facts
b	Counting techniques
c	Inverse operation
d	Guess and check
e	Cover up
f	Working backwards then guess and check
g	Working backwards then known fact
h	Working backwards
i	Transformations / equation as object

Data Analysis

In Rasch models, the probability of a specified response (i.e. right/wrong answer) is modeled as a logistic function of the difference between the person and item parameter. Before applying this model to the data it was therefore necessary to ascertain that the variable of item difficulty was uni-dimensional. Factor analysis was initially employed to verify that a one-factor model was an adequate fit to the data. Following this Rasch analysis was used to determine item difficulty and student ability. These scores were then related to the strategies that individual students used for each question.

Results

Item Difficulty

There was a huge variation between students in the number of equations that they were able to solve, with some questions being much harder than others (See Table 2).

Table 2
Item Difficulty

Equation	Number of Students with Correct Responses	Percentage of Students with Correct Responses	Rasch Score (Item Difficulty)
$n - 3 = 12$	565	91	-3.75
$18 = 3n$	511	82	-2.91
$n + 46 = 113$	523	84	-3.054
$\frac{n}{20} = 5$	206	33	1.111
$4n + 9 = 37$	400	64	-1.405
$3n - 8 = 19$	382	62	-1.153
$26 = 10 + 4n$	362	58	-0.883
$\frac{n+12}{4} = 18$	185	30	1.411
$5n + 70 = 150$	283	46	0.185
$2 + \frac{n}{4} = 8$	153	25	1.873
$5n - 2 = 3n + 6$	109	18	2.581
$2n - 3 = \frac{2n + 17}{5}$	24	4	5.064
$v = u + at$	5	1	7.283

In general one-step equations were easier to solve than two-step, which in turn were easier than equations with unknowns on both sides. However it should also be noted that equations involving division were harder than similar equations with other operators. Nevertheless one-step equations involving division were easier than two-step equations involving division.

Strategies Used

Some strategies (formal operations, guess and check) could be used to solve any equation, while others (e.g. inverse operation for one-step equations, working backwards for two-step equations) could be used for only a limited number of equations. For every equation (except the final one) there was a range of strategies successfully used by students, but the distribution of strategies varied from question to question. Responses to three questions are shown if Figure 2 to illustrate the ranges of strategies used.

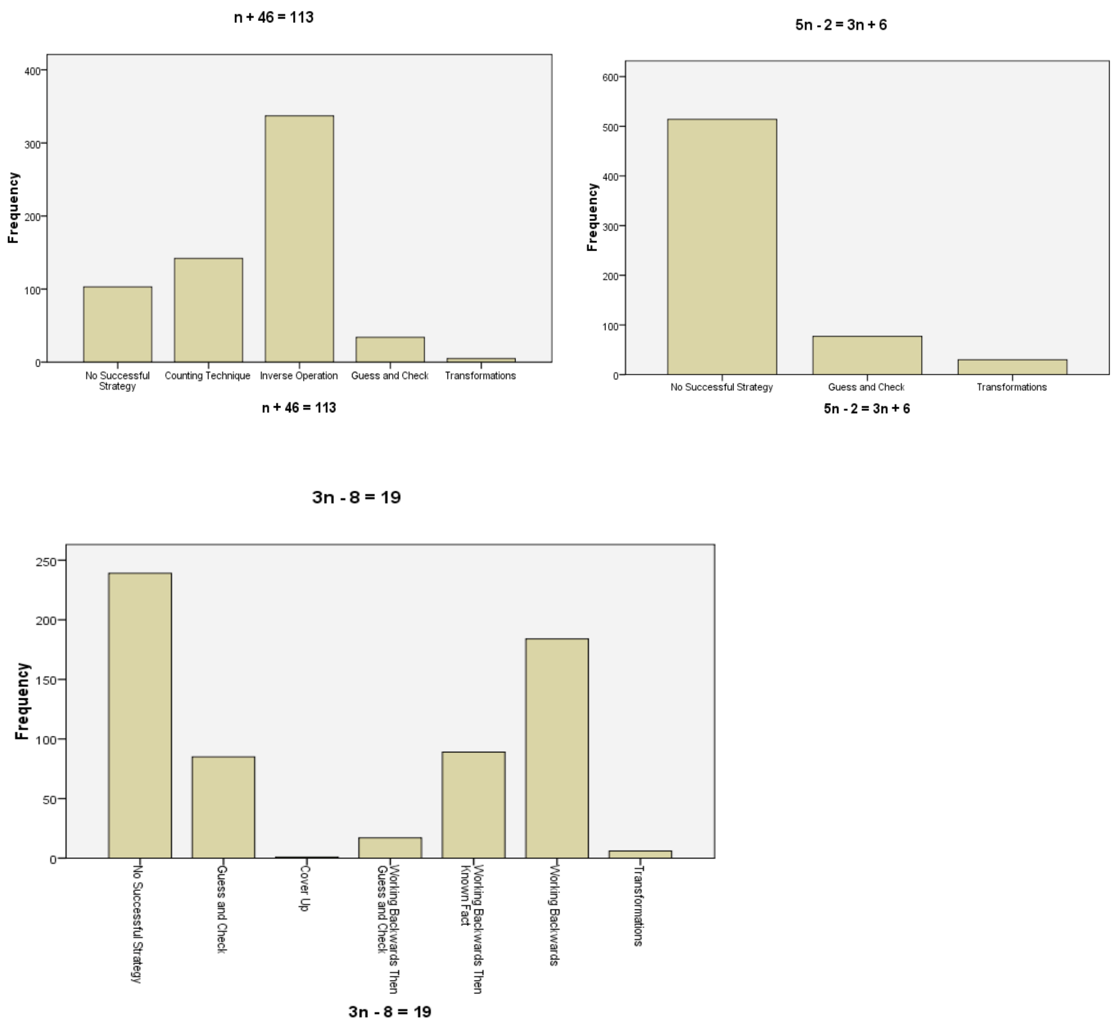

Figure 2. Students' strategies for three equations.

Hierarchy of Strategies

To establish a hierarchy of strategies was not straightforward as the pattern of strategy use varied from equation to equation, with some equations lending themselves to being solved by one strategy rather than another. Another difficulty was that able students often reverted to guess and check for difficult questions, even though they used other strategies for easier equations. Less able students, in contrast, used guess and check for easy equations and were unable to solve more difficult equations by any strategy.

The approach used, therefore, was to examine the strategies used on a question by question basis. For each question the ability of students using a particular strategy was investigated. Figure 3 shows the results for the same three questions shown in Figure 2.

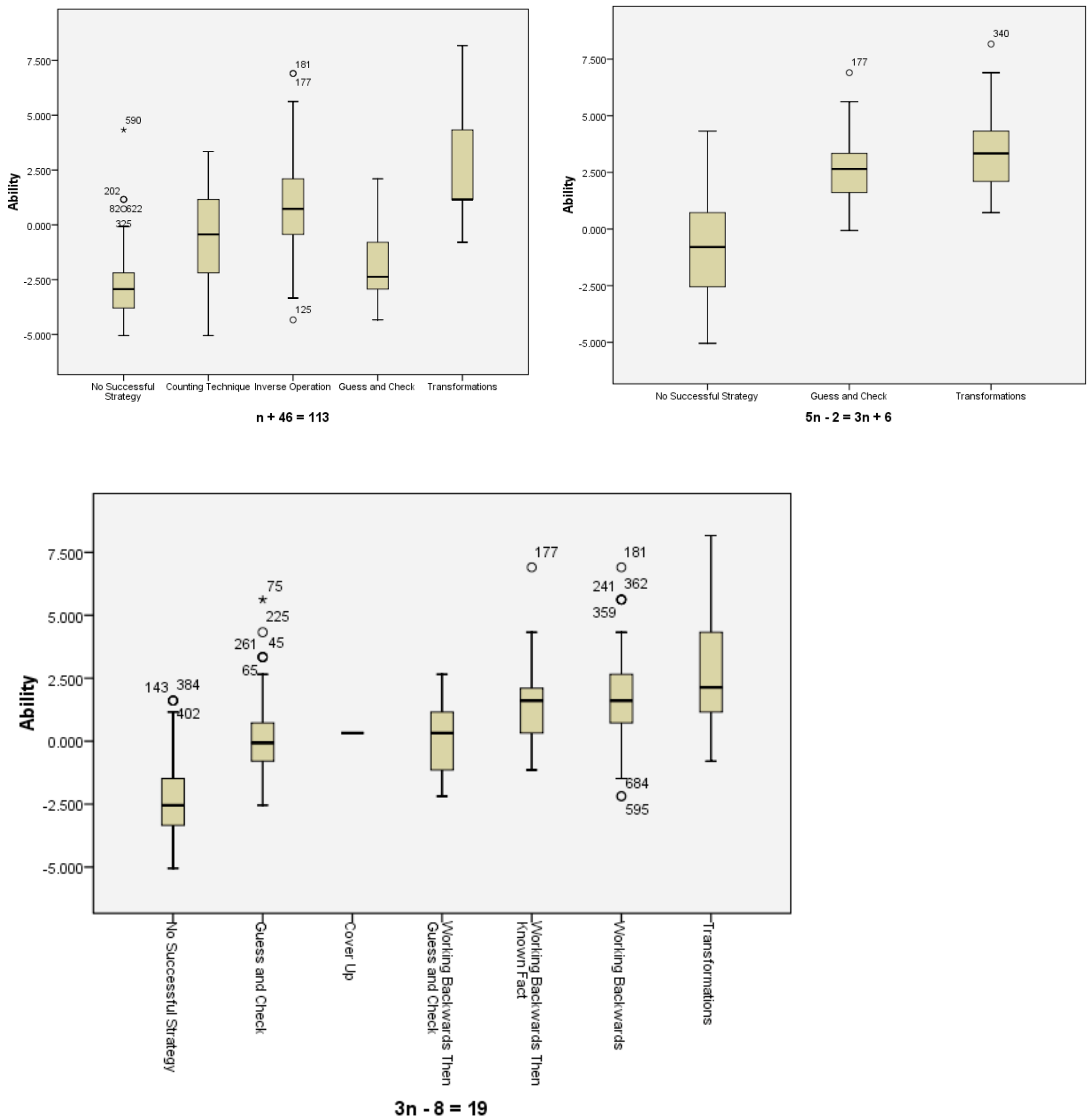

Figure 3. Ability of students using each strategy.

For each equation it was then possible to place in order the sophistication of strategies used. For example the equation $5n - 2 = 3n + 6$ was solved using either guess and check or transformations. The mean ability of students using transformations was higher than that of students using guess and check, indicating that transformations was the more sophisticated strategy.

The picture that emerged using this approach was fairly self-consistent. For all equations solved transformations was used by the most able students and guess and check by the least able.

For one-step equations it was not possible to discern between counting strategies and known basic facts because for any one equation both strategies were never used. However for three of the four one-step equations inverse operations were used by more able students than students using either counting strategies or known basic facts. The exception was $\frac{n}{20} = 5$, which was far more difficult than the other one-step equations. For this equation the most able students solved it using a known basic fact.

Cover up was used by such a small number of students that no clear relationship to the other strategies emerged.

For five of the six two-step equations working backwards was used by more able students than those using working backwards then known fact, which in turn was used by more able students than those using working backwards then guess and check. The exception was $\frac{n+12}{4} = 18$, but the number of students using any strategy other than working backwards was too small to draw any conclusions.

The strategies used only on one-step equations clearly could not be compared directly with those used only on two-step equations. However two-step equations were much harder than one-step and working backwards involves using inverse operations.

The order of sophistication of strategies indicated by this analysis is therefore guess and check, counting strategies / known basic facts, inverse operations, working backwards then guess and check, working backwards then known fact, working backwards, transformations.

Discussion and Conclusions

The strategies used by nearly all students varied from equation to equation. Most students chose a simple strategy that was sufficient for solving a particular equation, rather than necessarily using the most sophisticated strategy they were capable of.

Not surprisingly, one-step equations were easier than those involving two or more steps. However this study has shown that the strategy of solving one-step equations by inverse operations is used by more able students than those who use either known basic facts or counting strategies. These strategies in turn were used by more able students than those who solved one-step equations using guess and check.

Inverse operations are clearly required for the strategy of working backwards on two-step equations. It is important for teachers to realise that success at solving one-step equations does not necessarily mean that students can use inverse operations or that those students are ready to attempt two-step equations. Of particular note was the finding that using an inverse operation to solve an equation involving division was very difficult for students.

Earlier work (Linsell et al., 2006) identified that the strategy of working backwards is not homogeneous. Many students are only just grasping this strategy and can use it only when the first step reveals a known basic fact to them for the next step. These students use the strategy of working backwards then known facts. Other students are prevented from fully using working backwards because of lack of knowledge of multiplication and division facts. These students use the strategy of working backwards then guess and check. This current study has confirmed these observations and shown that the mean ability of students using these three strategies differed. Again it is important for teachers to appreciate that success at solving two-step equations does not necessarily imply that students fully understand the strategy of working backwards. In fact some students can solve two-step equations only by guess and check.

The strategy of transformations was used by only the most able students, with many students reverting to guess and check for equations with unknowns on both sides. As has been clearly identified (Herscovics & Linchevski, 1994), students have great difficulty with transformations.

Consistent with the perspective of Filloy and Sutherland (1996), it is suggested that these strategies are not simply alternative approaches to solving equations, but represent different stages of conceptual development. The approach used in this study is very similar to that used in the New Zealand Numeracy Development Projects, with strategy being separated out from knowledge required for strategy use. This approach allows the classification of students according to their most sophisticated strategy rather than the most difficult equation they are able to solve. Within numeracy teaching, students are grouped for instruction according to their most sophisticated strategy. It is suggested that a similar approach to grouping students is likely to be beneficial for teaching students to solve equations.

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